Exercise 62

Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 12.

$$y = x^3(x+2)^2(x-1)$$

Solution

To find the y-intercept, plug in $x = 0$ to the function.

$$y = (0)^3(0+2)^2(0-1) = 0$$

Therefore, the y-intercept is $(0,0)$. To find the x-intercept(s), set $y = 0$ and solve the equation for x.

$$x^3(x+2)^2(x-1) = 0$$

$$x = 0 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 1$$

Therefore, the x-intercepts are $(0,0)$ and $(-2,0)$ and $(1,0)$. Calculate the limit of the function as $x \to \pm\infty$.

$$\begin{aligned}
\lim_{x\to\infty} y &= \lim_{x\to\infty} x^3(x+2)^2(x-1) \\
&= \lim_{x\to\infty} x^3 \cdot x^2 \left(1 + \frac{2}{x}\right)^2 \cdot x \left(1 - \frac{1}{x}\right) \\
&= \lim_{x\to\infty} x^6 \left(1 + \frac{2}{x}\right)^2 \left(1 - \frac{1}{x}\right) \\
&= \lim_{x\to\infty} \frac{\left(1 + \frac{2}{x}\right)^2 \left(1 - \frac{1}{x}\right)}{\frac{1}{x^6}} \\
&= \frac{(1+0)^2(1-0)}{0} \\
&= \infty
\end{aligned}$$

In the second limit, make the substitution, $u = -x$, so that as $x \to -\infty$, $u \to \infty$.

$$\begin{aligned}
\lim_{x\to-\infty} y &= \lim_{u\to\infty} (-u)^3(-u+2)^2(-u-1) \\
&= \lim_{u\to\infty} -u^3 \cdot u^2 \left(-1+\frac{2}{u}\right)^2 \cdot u\left(-1-\frac{1}{u}\right) \\
&= \lim_{u\to\infty} u^6 \left(-1+\frac{2}{u}\right)^2 \left(1+\frac{1}{u}\right) \\
&= \lim_{u\to\infty} \frac{\left(-1+\frac{2}{u}\right)^2\left(1+\frac{1}{u}\right)}{\frac{1}{u^6}} \\
&= \frac{(-1+0)^2(1+0)}{0} \\
&= \infty
\end{aligned}$$

Below is a graph of the function versus x.